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The number of bound states for a discrete Schrödinger operator on $\mathbb{Z}^{\mathrm{N}}$, $N \geqslant 1$, lattices

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Abstract

We consider the discrete Schrödinger operator $-\Delta_d + \mathcal{U}$ in $\mathbb{Z}^N$, $N \geqslant 1$ in the case of a potential with negative part in an appropriate ℓ^σ-space (decays with an appropriate rate). We present a discrete analog of the method of Li and Yau (1983 *Commun. Math. Phys.* **88** 309–18), proving an explicit upper estimate on the number of bound states $\mathcal{N}_d(0) = \#\{j : \mu_j \leqslant 0\}$, which is independent of the dimension of the lattice. This is a major difference with the continuous counterpart estimate, which is not valid when $N = 1, 2$. As a consequence, a dimension-independent smallness criterion for the existence of bound states is derived in contrast to the continuous case as well as to the discrete case of vanishing potential. A short comment is made on possible applications of the results to the study of the dynamics of some particular spatially discrete nonlinear systems.

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1. Introduction

Consider the Schrödinger operator $-\Delta + \mathcal{U}(x)$, $x \in \mathbb{R}^N$, where $\mathcal{U}(x)$ is a rapidly decaying potential. A standing wave $\Phi(x,t) = \mathrm{e}^{-\mathrm{i}\lambda t}\Psi(x)$, $\lambda \in \mathbb{R}$, is a solution of the time-dependent Schrödinger equation

$$\mathrm{i}\partial_t \Phi = -\Delta\Phi + \mathcal{U}(x)\Phi, \qquad x \in \mathbb{R}^N.$$

Clearly Ψ and λ satisfy

$$-\Delta\Psi + \mathcal{U}(x)\Psi = \lambda\Psi, \qquad x \in \mathbb{R}^N, \tag{1.1}$$

i.e. $\Psi(x)$ and λ are an eigenfunction (*eigenstate*) and an eigenvalue of the operator $-\Delta + \mathcal{U}(x)$, respectively. Let the potential $\mathcal{U} : \mathbb{R}^N \to \mathbb{R}$, $N \geqslant 3$, and denote by $\mathcal{U}_-(x)$ its negative part. If $\int_{\mathbb{R}^N} |\mathcal{U}_-(x)|\,\mathrm{d}x < \infty$ then the number of non-positive eigenvalues $\mathcal{N}(0)$ of the problem (1.1) is finite (the operator $-\Delta + \mathcal{U}(x)$ has a discrete spectrum on the negative real line).

For potentials whose negative part is in $L^{N/2}(\mathbb{R}^N)$, Li and Yau [26] proved that the number $\mathcal{N}(0) = \#\{j : \lambda_j \leqslant 0\}$, (known as the number of bound states), can be estimated as

$$\mathcal{N}(0)\left(\frac{N(N-2)}{4\mathrm{e}}\right)^{\frac{N}{2}} \omega_{N-1} \leqslant \int_{\mathbb{R}^N} |\mathcal{U}_-|^{\frac{N}{2}}\,\mathrm{d}x, \qquad N \geqslant 3, \tag{1.2}$$

where ω_{N-1} is the volume of the unit $(N-1)$-sphere. The estimate (1.2) is also known as the Cwiekel–Lieb–Rosenbljum (CLR) inequality [11, 27, 28, 32] and was conjectured by Simon [33]. Inequality (1.2) has an important consequence: *if* $\|\mathcal{U}_-\|_{L^{\frac{N}{2}}(\mathbb{R}^N)} \ll 1$, *there are no negative eigenvalues for (1.1).*

The method of [26] (which contains brief historical notes on the problem) has been remarked for its simplicity. The main argument is based on the consideration of the auxiliary operator $\frac{-\Delta}{q(x)}$, with $q(x) > 0$ be in $L^{\frac{N}{2}}(\mathbb{R}^N)$. The auxiliary operator has a discrete spectrum in the positive real line. If μ_n denotes its nth eigenvalue, the function $\sum_{n=1}^{\infty}\exp(-2\mu_n t)$ is estimated with the help of the Sobolev inequality and its optimal constant,

$$\int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x \geqslant \frac{N(N-2)}{4}\omega_{N-1}^{\frac{2}{N}}\left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}\,\mathrm{d}x\right)^{\frac{N-2}{N}}, \qquad N \geqslant 3. \tag{1.3}$$

A series of reduction arguments combined with the fact that $\mathcal{N}(0)$ actually equals the number of eigenvalues less than 1 of $\frac{-\Delta}{q(x)}$, yields (1.2).

Now let N be a positive integer and $n := (n_1, n_2, \ldots, n_N) \in \mathbb{Z}^N$. In this paper, we consider the discrete eigenvalue problem

$$-(\Delta_d\psi)_n + \mathcal{U}_n\psi_n = \mu\psi_n, \qquad n \in \mathbb{Z}^N. \tag{1.4}$$

Here $\psi = \{\psi_n\}_{n\in\mathbb{Z}^N}$, and Δ_d stands for the N-dimensional discrete Laplacian

$$(\Delta_d\psi)_n = \sum_{m\in\mathcal{N}_n} \psi_m - 2N\psi_n, \tag{1.5}$$

where $\mathcal{N}_n$ denotes the set of $2N$ nearest neighbors of the point in $\mathbb{Z}^N$ with label n. This time, and in similarity with the continuous counterpart (1.1),

$$\phi_n(t) = \mathrm{e}^{-\mathrm{i}\mu t}\psi_n, \tag{1.6}$$

is a discrete standing wave of the time-dependent *discrete Schrödinger equation*

$$\mathrm{i}\partial_t\phi_n = -(\Delta_d\phi)_n + \mathcal{U}_n\phi_n, \qquad n \in \mathbb{Z}^N. \tag{1.7}$$

Note that solutions (1.6) fulfilling $|\psi_n| \to 0$ as $|n| \to \infty$ (here $|n| = \max_{1\leqslant i\leqslant N}|n_i|$ for $n = (n_1, n_2, \ldots, n_N) \in \mathbb{Z}^N$), are also known as *discrete breathers*. The eigenvalue problem (1.4) will be supplemented with this boundary condition at infinity, which can be formulated by considering (1.4) in appropriate sequence spaces.

In this work, we present a discrete analog of the proof of [26], estimating the number of discrete bound states

$$\mathcal{N}_d(0) = \#\{j : \mu_j \leqslant 0\},$$

assuming that the potential $\mathcal{U}(n) = \{\mathcal{U}_n\}_{n\in\mathbb{Z}^N}$ satisfies the condition $(P)\mathcal{U} = V + U$, where $V = \{V_n\}_{n\in\mathbb{Z}^N} \in \ell^\infty$ satisfies $V_n \geqslant \nu_0 > 0$ for all $n \in \mathbb{Z}^N$ and $U = \{U_n\}_{n\in\mathbb{Z}^N} \in \ell^{\frac{q-1}{q-2}}$ for some fixed $q > 2$ with $U_n < 0$ for all $n \in \mathbb{Z}^N$.

Condition (P) implies that the negative part of the sign-changing potential $\mathcal{U}$ *decays at an appropriate rate. The positive part is not decaying. It is only assumed to be bounded but not necessarily convergent as* $|n| \to \infty$.

For a deep analysis on the role of the rate of decay of *decaying potentials in discrete Schrödinger operators* in the case $N = 1$, we refer to the works of Damanik and Teschl

[12] and Damanik, Hundertmark, Killip and Simon [14]. Let us note first that for decaying potentials $\mathcal{U}_n$, it is known that when $\liminf_{|n|\to\infty} |n\mathcal{U}_n| > 1$, there are infinitely many bound states (cf [14]), independently of the sign of the decaying potential $\mathcal{U}_n$. Thus inverse square decay is critical for the existence of infinitely many bound states. Furthermore, in the case where $\mathcal{U}_n = -cn^{-2}$, it is known that the discrete spectrum below zero is finite when $c \leqslant 1/4$ and infinite when $c > 1/4$. In [12], the borderline behavior between the subcritical and supercritical cases was thoroughly studied.

The results of this paper do not only demonstrate the power and generality of the method of [26], but also some important differences with the continuous counterpart as well as with the discrete case but of decaying potentials. Let us first remark that here there is an analogy with [26], since under the sign-changing condition (P), we are always in the case of a *finite number of non-positive eigenvalues*. On the other hand, a first interesting observation is that in comparison with the continuous counterpart (1.2), the estimate on $\mathcal{N}_d(0)$ *is independent of the dimension of the* $\mathbb{Z}^N$*-lattice.* In theorem 2.5 it is verified that

$$\mathcal{N}_d(0) \leqslant \left(\frac{e}{\nu_0}\right)^{\frac{q-1}{q-2}} \sum_{n\in\mathbb{Z}^N} |U_n|^{\frac{q-1}{q-2}}. \tag{1.8}$$

As in the continuous CLR inequality (1.2), it follows from (1.8) that if $\|U_n\|_{\frac{q-1}{q-2}} \ll 1$, *there are no negative eigenvalues for (1.4), and this happens independently of the dimension of the lattice. To emphasize this major difference with the continuous case, we note that the continuous CLR inequality is not valid for* $N = 1, 2$. A dimension-dependent consequence similar to that of the continuous CLR inequality *holds for discrete operators only in the case of vanishing potentials*. As has been reported in [35], there exists $\epsilon > 0$ such that if $\mathcal{U}_n$ is a decaying potential, $N \geqslant 4$ and $\|\mathcal{U}_n\|_{\frac{N}{3}} \leqslant \epsilon$ the eigenvalue problem (1.4) has no solution.

The above corollary is another manifestation of the role of discreteness appearing this time from the study of the *linear* eigenvalue problem (1.4). It comes out through a simple dimension-independent 'discrete Sobolev inequality'. The role of discreteness and the main differences with the continuous case in the study of *nonlinear* localized modes for the *discrete nonlinear Schrödinger equations (DNLS)*, were first emphasized and proved by Weinstein in [37], through the study of a discrete analog of the Gagliardo–Nirenberg interpolation inequality (remark 2.8). See also [38] for a comparison with the continuous case.

The simple example we consider for the case $N = 1$ and of a negative part $U_n = -cn^{-1}$ (remark 2.7), which still generates finitely many eigenvalues below zero, demonstrates another main difference with the case of (strictly negative) decaying potentials. In the latter, it is known that when the whole potential is $\mathcal{U}_n = -cn^{-2+\epsilon}\epsilon > 0$, the eigenvalue problem (1.4) has infinitely many eigenvalues [12]. It is also important to note that under condition (P) and for a negative part $U_n = -cn^{-2}$, the notion of criticality with respect to the constant c is irrelevant. *This seems to be also in contrast with the continuous Schrödinger operator* $-\Delta + \mathcal{U}(x)$ *in the case where the potential (or its negative part) is the singular inverse square potential* $-c|x|^{-2}$. The known results are limited in the case of bounded domains $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, where the situation may drastically differ, depending on the critical value $c^* = (N-2)^2/4$—the optimal constant of the Hardy inequality. We refer to [6, 8, 9, 18] as well as to the recent work [22].

Let us finally remark on an important similarity to the condition (P) on the potential $\mathcal{U}(n) = \{\mathcal{U}_n\}_{n\in\mathbb{Z}^N}$ with the assumptions of [26] on the potential $\mathcal{U}(x)$, $x \in \mathbb{R}^N$ of (1.1). As in [26], we do not assume any particular dependence of the potential $\mathcal{U}_n$ on the spatial coordinates (the potential may depend on all or less spatial coordinates e.g., *the dimension of the potential may be different than the full-spatial dimension*). Thus we may conclude that *while both bounds (1.2) and the discrete analog (1.8) are independent of the dimension of the potential,* *only the discrete analog is, in addition, independent of the spatial dimension*. It is worth emphasizing that the dimension of the potential is of mathematical and physical significance. See Baizakov, Malomed and Salerno [3], and Sivan, Fibich and Weinstein [34] for the effect of the potential dimension in *DNLS lattices*. In fact, potentials whose dimension is smaller than the full-spatial dimension of the lattice may still support stable single-and multiple-peaked solitons in this lattice, [3]. The dimension of the potential has important effects on the stability of these solitons, [34].

We claim that the results of this paper could be of interest not only due to the importance of the discrete Schrödinger operator itself (see Damanik, Killip and Simon [13], Damanik, Hundertmark, Killip and Simon [14]) but also due to some further applications, especially regarding the dynamics of some nonlinear spatially discrete systems. Weyl's type estimate on the eigenvalues of the auxiliary discrete operator could be useful for the derivation of explicit estimates on the Hausdorff dimension of the global attractor associated with damped DNLS. A particular example concerning the dynamics of inhomogeneous waveguide arrays is briefly discussed in section 2.1. We refer to Eilbeck [16] and Kevrekidis, Rasmussen and Bishop [25] for reviews on DNLS equations and to [23, 24] for the existence of attractors for damped DNLS lattices. We also refer to Aceves, Luther, de Angelis, Rubenchik, Turitsyn [2, 37], for combinations of continuous and discrete nonlinear problems.

2. Estimation of the number of bound states $\mathcal{N}_d(0)$

We start with some preliminaries. First, we recall that between the standard sequence spaces

$$\ell^p = \left\{ \phi = \{\phi_n\}_{n\in\mathbb{Z}^N} : \|\phi\|_p = \left(\sum_{n\in\mathbb{Z}^N} |\phi_n|^p \right)^{\frac{1}{p}} < \infty \right\}, \tag{2.1}$$

the following elementary embedding relation [31] holds:

$$\ell^q \subset \ell^p, \qquad \|\phi\|_p \leqslant \|\phi\|_q 1 \leqslant q \leqslant p \leqslant \infty. \tag{2.2}$$

Note that the contrary holds for the $L^p(\Omega)$-spaces if $\Omega \subset \mathbb{R}^N$ has a finite measure. For $p = 2$, we get the usual Hilbert space of square-summable sequences endowed with the inner product

$$(\phi, \psi)_2 = \sum_{n\in\mathbb{Z}^N} \phi_n \psi_n, \quad \phi, \psi \in \ell^2. \tag{2.3}$$

We shall make thorough use of weighted sequence spaces: for a sequence $W = \{W_n\}_{n\in\mathbb{Z}^N}$ with $W_n > 0$ for all $n \in \mathbb{Z}^N$, we consider the weight $W^{-1} := \left\{\frac{1}{W_n}\right\}_{n\in\mathbb{Z}^N}$ and define the weighted sequence space $\ell^2_{W^{-1}}$ as

$$\ell^2_{W^{-1}} = \left\{ \phi = \{\phi_n\}_{n\in\mathbb{Z}^N}, : \|\phi\|_{\ell^2_{W^{-1}}} = \left(\sum_{n\in\mathbb{Z}^N} \frac{1}{W_n} |\phi_n|^2 \right)^{\frac{1}{2}} < \infty \right\}, \tag{2.4}$$

which is a Hilbert space endowed with the scalar product

$$(\phi, \psi)_{\ell^2_{W^{-1}}} = \sum_{n\in\mathbb{Z}^N} \frac{1}{W_n} \phi_n \psi_n, \qquad \phi, \psi \in \ell^2_{W^{-1}}. \tag{2.5}$$

Let us assume in addition that $W \in \ell^\infty$. Then

$$\sum_{n\in\mathbb{Z}^N} W_n |\phi_n|^2 \leqslant \|W\|^2_\infty \sum_{n\in\mathbb{Z}^N} \frac{1}{W_n} |\phi_n|^2, \qquad \text{for all} \quad \phi \in \ell^2_{W^{-1}}. \tag{2.6}$$

Definition 2.1. *If $W \in \ell^\infty$, we denote by ℓ^2_W the completion of the space $\ell^2_{W^{-1}}$ with respect to the norm*

$$\|\phi\|^2_{\ell^2_W} = \sum_{n\in\mathbb{Z}^N} W_n|\phi_n|^2. \tag{2.7}$$

As it can be easily noted,

$$\sum_{n\in\mathbb{Z}^N} |\phi_n|^2 \leqslant \|W\|_\infty \sum_{n\in\mathbb{Z}^N} \frac{1}{W_n}|\phi_n|^2, \qquad \text{for all} \quad \phi \in \ell^2_{W^{-1}}. \tag{2.8}$$

Thus, from definition 2.1 and (2.6)–(2.8) we have

$$\ell^2_{W^{-1}} \subset \ell^2 \subset \ell^2_W, \tag{2.9}$$

the inclusions being dense. We remark that if $W \in \ell^\sigma, \sigma \geqslant 1$, the space $\ell^2_{W^{-1}}$ $\left(\ell^2_W\right)$ contains sequences decaying faster (or slower) than the elements of ℓ^2. In what follows, we shall use the following compactness lemma.

Lemma 2.2 ([21, lemma 2.1, pg 118]**.** *We assume that the positive sequence $W \in \ell^{\frac{q-1}{q-2}}$ for some $q > 2$ such that $W_n > 0$ for all $n \in \mathbb{Z}^N$. Then $\ell^2 \subset \ell^2_W$ with a compact inclusion.*

At this point, we mention that under condition (P), the operator $-\Delta_d + \mathcal{U}$ has a discrete spectrum in the negative real line: consider the associated quadratic form $\delta_d : \ell^2 \times \ell^2 \to \mathbb{R}$,

$$\delta_d(\phi, \psi) := (-\Delta_d\phi, \psi)_2 + \sum_{n\in\mathbb{Z}^N} \mathcal{U}_n\phi_n\psi_n, \qquad \phi, \psi \in \ell^2.$$

It is a Garding form, since due to (P) and (2.16),

$$\delta_d(\psi, \psi) = (-\Delta_d\psi, \psi)_2 + \sum_{n\in\mathbb{Z}^N} \mathcal{U}_n|\psi_n|^2 \geqslant \nu_0\|\psi\|^2_2 - \sum_{n\in\mathbb{Z}^N} |U_n||\psi_n|^2,$$

and by lemma (2.2), the embedding $\ell^2 \subset \ell^2_W$ is compact for $W = |U|$. Hence, by [39, theorem 22.G, pg 369], $-\Delta_d + \mathcal{U}$ has infinitely many eigenvalues which if counted according to their multiplicity,

$$-1 < \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_j \leqslant \cdots \to \infty, \qquad \text{as} \quad j \to \infty.$$

The smallest eigenvalue μ_1 can be characterized by the minimization problem

$$\mu_1 = \inf\{\delta_d(\psi, \psi) : \|\psi\|_{\ell^2_W} = 1\}, \qquad W = |U|. \tag{2.10}$$

If $\psi_1 = \{\psi_{1,n}\}_{n\in\mathbb{Z}^N}$ is the eigenstate associated with μ_1, then $\phi_n(t) = \mathrm{e}^{-\mathrm{i}\mu_1 t}\psi_{1,n}$ is the *ground state breather solution* of (1.7). A key work on the existence and thresholds for the existence of nonlinear localized modes for DNLS lattices is that of Weinstein [37]. We also refer to the recent work [10] for the DNLS with saturable and power nonlinearity.

The estimation of $\mathcal{N}_d(0)$ will be based on a Weyl's type estimate on the eigenvalues of the auxiliary problem

$$-(\Delta_d\psi)_n + V_n\psi_n = \lambda W_n\psi_n, \qquad n \in \mathbb{Z}^N. \tag{2.11}$$

For V being as in condition (P) and W as in lemma 2.2, we consider the discrete operator $\mathcal{L}_0 : \ell^2_{W^{-1}} \to \ell^2_W$,

$$(\mathcal{L}_0\psi)_{n\in\mathbb{Z}^N} = \frac{1}{W_n}\left[-(\Delta_d\psi)_n + V_n\psi_n\right], \tag{2.12}$$

which is well defined and continuous due to

$$\|\mathcal{L}_0\psi\|^2_{\ell^2_W} \leqslant (4N + c_1)\|\psi\|^2_{\ell^2_{W^{-1}}}, \qquad c_1 = \|V\|_\infty\|W\|^2_\infty, \qquad \text{for all} \quad \psi \in \ell^2_{W^{-1}}.$$

Proposition 2.3. *Assume that (a)* $W \in \ell^{\frac{q-1}{q-2}}$ *for some fixed* $q > 2$, $W_n > 0$ *for all* $n \in \mathbb{Z}^N$, *and (b)* $V \in \ell^\infty$, $V_n \geqslant \nu_0 > 0$ *for all* $n \in \mathbb{Z}$. *The operator* $\mathcal{L}_0 : \ell^2_{W^{-1}} \to \ell^2_W$ *has an extension* $\mathcal{L} : D(\mathcal{L}) \subseteq \ell^2_W \to \ell^2_W$ *which is a non-negative and self-adjoint operator. Moreover, there exists a complete orthonormal basis* $\psi_j = \{\psi_{j,n}\}_{n\in\mathbb{Z}^N}$, $j = 1, 2, \ldots,$ *of* ℓ^2_W *consisting of eigenvectors of* $\mathcal{L}$ *with the eigenvalue sequence*

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots \to \infty, \qquad as \quad j \to \infty. \tag{2.13}$$

Proof. For any $\phi, \psi \in \ell^2_{W^{-1}}$,

$$\begin{aligned}(\mathcal{L}\psi, \phi)_{\ell^2_W} &= (-\Delta_d\psi, \phi)_2 + \sum_{n\in\mathbb{Z}^N} V_n\psi_n\phi_n \\ &= (-\Delta_d\phi, \psi)_2 + \sum_{n\in\mathbb{Z}^N} V_n\phi_n\psi_n = (\mathcal{L}\phi, \psi)_{\ell^2_W},\end{aligned} \tag{2.14}$$

due to the symmetry of $-\Delta_d$ in the ℓ^2-scalar product. Furthermore,

$$(\mathcal{L}\psi, \psi)_{\ell^2_W} = (-\Delta_d\psi, \psi)_2 + \sum_{n\in\mathbb{Z}^N} V_n|\psi_n|^2. \tag{2.15}$$

From the right-hand side of (2.15), we may define a norm on ℓ^2

$$\|\psi\|^2_{2,V} := (-\Delta_d\psi, \psi)_2 + \sum_{n\in\mathbb{Z}^N} V_n|\psi_n|^2.$$

The norm $\|\cdot\|_{2,V}$ is equivalent to the standard norm of ℓ^2 since

$$\nu_0\|\psi\|^2_2 \leqslant \|\psi\|^2_{2,V} \leqslant (2N + c_2)\|\psi\|^2_2, \qquad c_2 = \|V\|_\infty. \tag{2.16}$$

Then, by using (2.9), (2.15) and (2.16), we deduce that

$$(\mathcal{L}\psi, \psi)_{\ell^2_W} \geqslant c\|\psi\|^2_{\ell^2_W}, \qquad \text{for all} \quad \psi \in \ell^2_{W^{-1}}, \tag{2.17}$$

with $c = \nu_0/\|W\|_\infty$. By the definitions 2.1 and (2.17), Friedrich's extension theory (cf Zeidler, [39, theorem 19.C, pg 126]) is applicable to the operator $\mathcal{L}_0$ with the domain of the definition $D(\mathcal{L}_0) = \ell^2_{W^{-1}}$. The energy space $\mathcal{X}_E$ is the completion of $\ell^2_{W^{-1}}$ in the norm $\|\cdot\|_{2,V}$ and is a Hilbert space with the inner product

$$[\phi, \psi] := (-\Delta\phi, \psi)_2 + \sum_{n\in\mathbb{Z}^N} V_n\phi_n\psi_n.$$

Due to the equivalence of norms (2.16), we have $\mathcal{X}_E \equiv \ell^2$. The Friedrich's extension of the operator $\mathcal{L}_0$ is the operator $\mathcal{L} : D(\mathcal{L}) \to \ell^2$ with its domain defined as

$$D(\mathcal{L}) := \left\{\psi \in \mathcal{X}_E \equiv \ell^2 : \frac{1}{W_n}[-(\Delta_d\psi)_n + V_n\psi_n] \in \ell^2_W\right\}. \tag{2.18}$$

Since ℓ^2 is compactly embedded in ℓ^2_W, there exists a complete orthonormal basis of ℓ^2_W consisting of the eigenvectors $\psi_j = \{\psi_{j,n}\}_{n\in\mathbb{Z}^N}$, $j = 1, 2, \ldots,$ of $\mathcal{L}$ with the eigenvalue sequence (2.13). □

We shall implement the method of [26], to prove a Weyl's type estimate on the eigenvalues of the problem (2.11). This estimate is given in

Theorem 2.4. *Assume that (a) $W \in \ell^{\frac{q-1}{q-2}}$ for some fixed $q > 2$, $W_n > 0$ for all $n \in \mathbb{Z}^N$ and (b) $V \in \ell^\infty$, $V_n \geqslant \nu_0 > 0$ for all $n \in \mathbb{Z}$. The eigenvalues (2.13) of the eigenvalue problem (2.11) satisfy Weyl's type estimate*

$$\lambda_j \geqslant \frac{\nu_0}{\mathrm{e}}\left(\sum_{n\in\mathbb{Z}^N} W_n^{\frac{q-1}{q-2}}\right)^{-\frac{q-2}{q-1}} j^{\frac{q-2}{q-1}}, \qquad j \to \infty. \tag{2.19}$$

Proof. The Friedrich's extension $\mathcal{L}$ being non-negative and self-adjoint in ℓ^2_W, gives rise to the semigroup of operators $\mathrm{e}^{-\mathcal{L}t}$ for every $t > 0$, possessing a kernel $\mathcal{H}(t) = \{\mathcal{H}_{m,n}(t)\}_{m,n\in\mathbb{Z}^N}$ such that $\mathcal{H}_{m,n}(t) > 0$ for all $(m, n, t) \in \mathbb{Z}^N \times \mathbb{Z}^N \times (0, \infty)$. The operator $\mathcal{L}$ has compact resolvent, thus $\mathcal{H}_{m,n}(t)$ can be represented as (15, cf Davies [pg 108])

$$\mathcal{H}_{m,n}(t) = \sum_{i=1}^{\infty} \exp(-\lambda_i t)\psi_{i,m}\psi_{i,n} \qquad \text{for all} \quad m, n \in \mathbb{Z}^N. \tag{2.20}$$

We quote that $\mathcal{H}_{m,n}(t)$ solves the problem

$$\begin{aligned}
&\partial_t \mathcal{H}_{m,n}(t) - \frac{1}{W_n}[(\Delta_d \mathcal{H}(t))_n - V_n \mathcal{H}_{m,n}(t)] = 0, \qquad \text{in} \quad \mathbb{Z}^N \times \mathbb{Z}^N \times (0,\infty),\\
&\mathcal{H}_{m,n}(t) > 0, \qquad \text{in} \quad \mathbb{Z}^N \times \mathbb{Z}^N \times (0,\infty),\\
&\lim_{|m||n|\to\infty} \mathcal{H}_{m,n}(t) = 0, \qquad t \in (0,\infty).
\end{aligned} \tag{2.21}$$

Since the eigenvectors ψ_j, $j = 1, 2, \ldots$, form an orthonormal basis of ℓ^2_W, i.e.,

$$\sum_{n\in\mathbb{Z}^N} W_n \psi_{i,n}\psi_{j,n} = \delta_{ij}, \tag{2.22}$$

it follows that

$$\mathcal{F}(t) := \sum_{i=1}^{\infty} \exp(-2\lambda_i t) = \sum_{m\in\mathbb{Z}^N}\sum_{n\in\mathbb{Z}^N} \mathcal{H}^2_{m,n}(t) W_m W_n. \tag{2.23}$$

Applying Hölder's inequality, we get the estimate

$$\begin{aligned}
\mathcal{F}(t) &= \sum_{m\in\mathbb{Z}^N} W_m \sum_{n\in\mathbb{Z}^N} \mathcal{H}_{m,n}^{\frac{2(q-1)}{2q-3}}(t)\mathcal{H}_{m,n}^{\frac{2(q-2)}{2q-3}}(t) W_n\\
&\leqslant \sum_{m\in\mathbb{Z}^N} W_m \left[\sum_{n\in\mathbb{Z}^N}\left(\mathcal{H}_{m,n}^{\frac{2(q-1)}{2q-3}}(t)\right)^{2q-3}\right]^{\frac{1}{2q-3}} \left[\sum_{n\in\mathbb{Z}^N}\left(\mathcal{H}_{m,n}^{\frac{2(q-2)}{2q-3}}(t) W_n\right)^{\frac{2q-3}{2(q-2)}}\right]^{\frac{2(q-2)}{2q-3}}\\
&= \sum_{m\in\mathbb{Z}^N} W_m \left(\sum_{n\in\mathbb{Z}^N} \mathcal{H}^{2(q-1)}_{m,n}(t)\right)^{\frac{1}{2q-3}} \left(\sum_{n\in\mathbb{Z}^N} \mathcal{H}_{m,n}(t) W_n^{\frac{2q-3}{2(q-2)}}\right)^{\frac{2(q-2)}{2q-3}}\\
&= \sum_{m\in\mathbb{Z}^N} W_m^{\frac{q-1}{2q-3}} W_m^{\frac{q-2}{2q-3}} \left(\sum_{n\in\mathbb{Z}^N} \mathcal{H}^{2(q-1)}_{m,n}(t)\right)^{\frac{1}{2q-3}} \left(\sum_{n\in\mathbb{Z}^N} \mathcal{H}_{m,n}(t) W_n^{\frac{2q-3}{2(q-2)}}\right)^{\frac{2(q-2)}{2q-3}}\\
&\leqslant \left[\sum_{m\in\mathbb{Z}^N} W_m \left(\sum_{n\in\mathbb{Z}^N} \mathcal{H}^{2(q-1)}_{m,n}(t)\right)^{\frac{1}{q-1}}\right]^{\frac{q-1}{2q-3}} \left[\sum_{m\in\mathbb{Z}^N} W_m \left(\sum_{n\in\mathbb{Z}^N} \mathcal{H}_{m,n}(t) W_n^{\frac{2q-3}{2(q-2)}}\right)^2\right]^{\frac{q-2}{2q-3}}.
\end{aligned} \tag{2.24}$$

We consider the function

$$\mathcal{Q}_m(t) = \sum_{n\in\mathbb{Z}^N} \mathcal{H}_{m,n}(t) W_n^{\frac{2q-3}{2(q-2)}}. \tag{2.25}$$

From (2.20) and (2.22), we deduce that $\mathcal{Q}_m(0) = W_m^{\frac{1}{2(q-2)}}$. Thus $\mathcal{Q}(t)$ is the solution of the linear lattice differential equation

$$\partial_t \mathcal{Q}_m(t) - \frac{1}{W_m}[(\Delta_d \mathcal{Q}(t))_m - V_m \mathcal{Q}_m(t)] = 0, \qquad \text{in} \quad \mathbb{Z}^N \times (0,\infty), \tag{2.26}$$

$$\begin{aligned} &\mathcal{Q}_m(0) = W_m^{\frac{1}{2(q-2)}}, && \text{in} \quad \mathbb{Z}^N \times (0,\infty), \\ &\lim_{|m|\to\infty} \mathcal{Q}_m(t) = 0, && t \in (0,\infty). \end{aligned} \tag{2.27}$$

Multiplying (2.26) by $\mathcal{Q}(t)$ in the ℓ^2_W-inner product, we get the energy equation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m\in\mathbb{Z}^N} W_m \mathcal{Q}_m^2(t) + (-\Delta_d \mathcal{Q}(t), \mathcal{Q}(t))_2 + \sum_{m\in\mathbb{Z}^N} V_m \mathcal{Q}_m^2(t) = 0.$$

This energy combined with (2.27), gives the estimate

$$\sum_{m\in\mathbb{Z}^N} W_m \mathcal{Q}_m^2(t) \leqslant \sum_{m\in\mathbb{Z}^N} W_m \mathcal{Q}_m^2(0) = \sum_{m\in\mathbb{Z}^N} W_m^{\frac{q-1}{q-2}}. \tag{2.28}$$

Now, by inserting (2.25) and (2.26) into (2.24), the inequality

$$\begin{aligned} \mathcal{F}(t) &\leqslant \left[\sum_{m\in\mathbb{Z}^N} W_m \left(\sum_{n\in\mathbb{Z}^N} \mathcal{H}_{m,n}^{2(q-1)}(t)\right)^{\frac{1}{q-1}}\right]^{\frac{q-1}{2q-3}} \left(\sum_{m\in\mathbb{Z}^N} W_m \mathcal{Q}_m^2(t)\right)^{\frac{q-2}{2q-3}} \\ &\leqslant \left[\sum_{m\in\mathbb{Z}^N} W_m \left(\sum_{n\in\mathbb{Z}^N} \mathcal{H}_{m,n}^{2(q-1)}(t)\right)^{\frac{1}{q-1}}\right]^{\frac{q-1}{2q-3}} \left(\sum_{m\in\mathbb{Z}^N} W_m \mathcal{Q}_m^2(0)\right)^{\frac{q-2}{2q-3}} \\ &\leqslant \left[\sum_{m\in\mathbb{Z}^N} W_m \left(\sum_{n\in\mathbb{Z}^N} \mathcal{H}_{m,n}^{2(q-1)}(t)\right)^{\frac{1}{q-1}}\right]^{\frac{q-1}{2q-3}} \left(\sum_{m\in\mathbb{Z}^N} W_m^{\frac{q-1}{q-2}}\right)^{\frac{q-2}{2q-3}}, \end{aligned}$$

follows. Hence, we have that

$$\mathcal{F}^{\frac{2q-3}{q-1}}(t) \leqslant \left[\sum_{m\in\mathbb{Z}^N} W_m \left(\sum_{n\in\mathbb{Z}^N} \mathcal{H}_{m,n}^{2(q-1)}(t)\right)^{\frac{1}{q-1}}\right] \left(\sum_{m\in\mathbb{Z}^N} W_m^{\frac{q-1}{q-2}}\right)^{\frac{q-2}{q-1}}. \tag{2.29}$$

To estimate the right-hand side of (2.29) further, we shall use the inequality

$$\left(\sum_{n\in\mathbb{Z}^N} \mathcal{H}_{m,n}^{2(q-1)}(t)\right)^{\frac{1}{q-1}} \leqslant \frac{1}{\nu_0}\left[(-\Delta_d \mathcal{H}(t), \mathcal{H}(t))_2 + \sum_{n\in\mathbb{Z}^N} V_n \mathcal{H}_{m,n}^2(t)\right]. \tag{2.30}$$

Inequality (2.30) comes from (2.2) and the equivalence of norms (2.16). On the other hand, from (2.21) we infer that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t) &= 2\sum_{m\in\mathbb{Z}^N}\sum_{n\in\mathbb{Z}^N}\mathcal{H}_{m,n}(t)W_mW_n\partial_t\mathcal{H}_{m,n}(t)\\
&= 2\sum_{m\in\mathbb{Z}^N}\sum_{n\in\mathbb{Z}^N}\mathcal{H}_{m,n}(t)W_mW_n\left\{-\frac{1}{W_n}\left[(\Delta_d\mathcal{H}(t))_n - V_n\mathcal{H}_{m,n}(t)\right]\right\}\\
&= -2\sum_{m\in\mathbb{Z}^N}W_m\left[(-\Delta_d\mathcal{H}(t),\mathcal{H}(t))_2 + \sum_{n\in\mathbb{Z}^N}V_n\mathcal{H}^2_{m,n}(t)\right].
\end{aligned} \tag{2.31}$$

Thus, by inserting (2.30) into (2.29) we have that

$$\mathcal{F}^{\frac{2q-3}{q-1}}(t) \leqslant \frac{1}{\nu_0}\left\{\sum_{m\in\mathbb{Z}^N}W_m\left[(-\Delta_d\mathcal{H}(t),\mathcal{H}(t))_2 + \sum_{n\in\mathbb{Z}^N}V_n\mathcal{H}^2_{m,n}(t)\right]\right\}\left(\sum_{m\in\mathbb{Z}^N}W_m^{\frac{q-1}{q-2}}\right)^{\frac{q-2}{q-1}},$$

which can be rewritten—by using (2.31)—as

$$\mathcal{F}^{\frac{2q-3}{q-1}}(t) \leqslant -\frac{1}{2\nu_0}\left(\sum_{m\in\mathbb{Z}^N}W_m^{\frac{q-1}{q-2}}\right)^{\frac{q-2}{q-1}}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t). \tag{2.32}$$

Integration of (2.32), with respect to time and (2.23), yields

$$\sum_{i=1}^{\infty}\exp(-2\lambda_i t) \leqslant \left[\frac{q-1}{2\nu_0(q-2)}\right]^{\frac{q-1}{q-2}}\left(\sum_{m\in\mathbb{Z}^N}W_m^{\frac{q-1}{q-2}}\right)^{\frac{q-2}{q-1}} t^{-\frac{q-1}{q-2}}. \tag{2.33}$$

Setting

$$t = \frac{q-1}{2(q-2)\lambda_j},$$

in (2.33), we conclude with the estimate

$$j\,\mathrm{e}^{-\frac{q-1}{q-2}} \leqslant \sum_{i=1}^{\infty}\exp\left(\frac{-\lambda_i(q-1)}{\lambda_j(q-2)}\right) \leqslant \nu_0^{-\frac{q-1}{q-2}}\left(\sum_{m\in\mathbb{Z}^N}W_m^{\frac{q-1}{q-2}}\right)\lambda_j^{\frac{q-1}{q-2}}. \tag{2.34}$$

From inequality (2.34), the estimate (2.19) readily follows. □

Using theorem 2.4 we may proceed to the estimation of $\mathcal{N}_d(0)$ as in [26].

Theorem 2.5. *Consider the eigenvalue problem (1.4) assuming that the potential $\mathcal{U}$ satisfies condition (P). The number of bound states $\mathcal{N}_d(0)$ satisfies the estimate (1.8).*

Proof. In this case, the quadratic form associated with the eigenvalue problem (1.4) is

$$\begin{aligned}
\frac{(-\Delta_d\psi,\psi)_2 + \sum_{n\in\mathbb{Z}^N}V_n|\psi_n|^2 + \sum_{n\in\mathbb{Z}^N}U_n|\psi_n|^2}{\sum_{n\in\mathbb{Z}^N}|\psi_n|^2} &= \frac{\sum_{n\in\mathbb{Z}^N}|U_n||\psi_n|^2}{\sum_{n\in\mathbb{Z}^N}|\psi_n|^2}\\
&\quad\times\left\{\frac{(-\Delta_d\psi,\psi)_2 + \sum_{n\in\mathbb{Z}^N}V_n|\psi_n|^2}{\sum_{n\in\mathbb{Z}^N}|U_n||\psi_n|^2} - 1\right\}.
\end{aligned} \tag{2.35}$$

Thus the subspace on which the left-hand side of (2.35) is non-positive has dimension equal to that of the subspace on which the quadratic form

$$\frac{(-\Delta_d\psi,\psi)_2+\sum_{n\in\mathbb{Z}^N}V_n|\psi_n|^2}{\sum_{n\in\mathbb{Z}^N}|U_n||\psi_n|^2} \tag{2.36}$$

is less than or equal to one. Since (2.36) is the quadratic form associated with the operator $\mathcal{L}: D(\mathcal{L})\subseteq \ell^2_W\to\ell^2_W$ in the case $W=|U|$, we have that the number of non-positive eigenvalues of the eigenvalue problem (1.4) is equal to the number of eigenvalues of the eigenvalue problem (2.11) which are less than or equal to 1.

Now let λ_j be the greatest eigenvalue of the eigenvalue problem (2.11) which is less than or equal to 1. Then due to the estimate (2.19) we have

$$\sum_{n\in\mathbb{Z}^N}|U_n|^{\frac{q-1}{q-2}}\geqslant\lambda_j^{\frac{q-1}{q-2}}\sum_{n\in\mathbb{Z}^N}|U_n|^{\frac{q-1}{q-2}}\geqslant j\left(\frac{\nu_0}{e}\right)^{\frac{q-1}{q-2}}\geqslant\mathcal{N}_d(0)\left(\frac{\nu_0}{e}\right)^{\frac{q-1}{q-2}},$$

which is (1.8). □

Theorem 2.5 implies the following

Corollary 2.6. *Consider the eigenvalue problem (1.4) under condition* (P)*. If the negative part satisfies* $\|U_n\|_{\frac{q-1}{q-2}}\ll 1$*, then there are no negative eigenvalues, for any* $N\geqslant 1$*.*

Remark 2.7. 1D example: as an example we consider the case $n\in\mathbb{Z}^+$ and $|U_n|=n^{-1}$, a case of (1.4) supplemented with the *one-sided Dirichlet boundary condition*, $\psi_0=0$ on $n=0$, [12]. Applying (1.8), one gets the estimate

$$\mathcal{N}_d(0)\leqslant\left(\frac{e}{\nu_0}\right)^{\frac{q-1}{q-2}}\zeta\left(\frac{q-1}{q-2}\right),\qquad\text{for all}\quad q\geqslant 3, \tag{2.37}$$

since $n^{-1}\in\ell^\sigma$ for all $\sigma>1$. Note that when $\mathcal{U}_n=-cn^{-2+\epsilon}\epsilon>0$ (strictly negative potential), there are infinitely many eigenvalues, [12].

Remark 2.8. The dimension-independent estimate (1.8) for the linear discrete operator (1.4) is another manifestation of the role of discreteness since in the continuous linear counterpart the number of bound states depends on the dimension due to the critical Sobolev inequality (1.3). For the discrete operator (1.4) the independence of the dimension is a consequence of the dimension-independent 'discrete Sobolev inequality' (2.30). The estimate (1.8) is valid for any dimension $N\geqslant 1$ in contrast to the continuous counterpart (1.2) which is not valid when $N=1,2$, and implies the dimension-independent criterion of corollary 2.6.

In the study of *nonlinear localized modes* on multidimensional lattices, an important difference compared with the continuous case has been rigorously proved for the DNLS equation in [37]. In [37], the hypothesis suggested by Flach, Kladko and MacKay [19] on excitation thresholds for the existence of nonlinear localized modes has been resolved. For the DNLS equation with power nonlinearity $F(z)=|z|^{2\sigma}z$, it was proved that an excitation threshold exists if $\sigma\geqslant 2/N$. For instance, the results of [37] prove that there exists an excitation threshold $\mathcal{R}_{\mathrm{thresh}}(N)$ on the power of periodic solutions, as well as the existence of a frequency $\omega^*>0$ on which this threshold value on the power is achieved. The corresponding solution $\psi_n(t)=\mathrm{e}^{\mathrm{i}\omega^*t}\phi_n$ is a ground state having power $\|\phi\|_2^2=\mathcal{R}_{\mathrm{thresh}}$—the excitation threshold value. This dramatically differs with the continuous case where an excitation threshold appears only in the case of the critical nonlinearity $\sigma=2/N$, [38]. The Weinstein's excitation threshold [37] is related to the best constant of a discrete analog of the Sobolev–Gagliardo–Nirenberg inequality

$$\sum_{n\in\mathbb{Z}^N} |\phi_n|^{2\sigma+2} \leqslant C \left(\sum_{n\in\mathbb{Z}^N} |\phi_n|^2\right)^{\sigma} (-\Delta_d\phi, \phi)_2, \qquad \sigma \geqslant 2/N. \tag{2.38}$$

When $\sigma < 2/N$ standing waves of arbitrary small power exist, [37]. However it was proved in [10] that there exist dimension-independent lower bounds for the power of standing wave solutions for the DNLS equation with power (as well as for the saturable) nonlinearity. These explicit lower bounds are different form the Weinstein's thresholds in the sense that *no periodic localized solution can have power less than the prescribed estimates of* [10] *even when* $\sigma < 2/N$. The 'global character' of the estimates of [10] is revealed when one considers 'limiting' cases of small (large) values of $\sigma < 2/N$ (large values of $\sigma \geqslant 2/N$—the case of excitation threshold), as it was justified by numerical simulations.

In a similar manner and motivated by the 'global character' of the dimension-independent estimates on the power of nonlinear localized modes of [10], we may think for a possible 'global character' of the estimate on the number of bound states (1.8) for the linear discrete operator (1.4). To this end, numerical simulations for testing (1.8) could be of interest.

2.1. Applications: dynamics of light in inhomogeneous waveguide arrays

We conclude with a short comment on some applications of the results to the study of the dynamics of some spatially discrete nonlinear systems. A particular example is a DNLS equation with *site-dependent coupling strength*

$$\mathrm{i}\partial_z A_n + \epsilon_{n,n+1} A_n + \epsilon_{n,n-1} A_{n-1} + |A_n|^2 A_n = 0, \qquad n \in \mathbb{Z}. \tag{2.39}$$

The DNLS system (2.39) describes the evolution of the slowly varying mode amplitude A_n along the propagation direction z in an inhomogeneous waveguide array (see Ablowitz and Musslimani [1], Pertsch, Peschel and Lederer [29, pg 746]). The case $\epsilon_{n,m} = \text{const}$, simplifies to the usual DNLS equation. The variation of the coupling must obey a symmetry relation $\epsilon_{n,m} = \epsilon_{m,n}$ to ensure energy conservation. Cases of interest include the periodic modulation of the coupling only along the direction z (e.g. $\epsilon_{n,n+1} = \epsilon_{n,n-1} \sim \cos(z)$) or only its linear variation in the transverse direction i.e. $\epsilon_{n,n+1} = \epsilon_{n,n-1} = \epsilon_n \sim n$. Although in the latter, various scalings can be applied (e.g. to transform (2.39) to the DNLS system with an unbounded potential, [29], Pertsch, Peschel and Lederer [30]), it is still convenient to keep the equation in the form (2.39): we remark that the transformation $A_n \to A_n\,\mathrm{e}^{\mathrm{i}z\epsilon_n}$ transforms the equation (2.39) to

$$\mathrm{i}\partial_z A_n + \epsilon_n (A_n - 2A_n + A_{n-1}) + |A_n|^2 A_n = 0, \qquad n \in \mathbb{Z}. \tag{2.40}$$

Aiming at the study, the physically justified case of a weakly damped and periodically forced version of (2.40),

$$\mathrm{i}\partial_z A_n + \mathrm{i}\delta A_n + \epsilon_n (A_n - 2A_n + A_{n-1}) + |A_n|^2 A_n = g_n\,\mathrm{e}^{\mathrm{i}\omega z}, \qquad n \in \mathbb{Z}, \quad \omega > 0, \tag{2.41}$$

we note that the transformation $A_n \to A_n\,\mathrm{e}^{\mathrm{i}\omega z}$ brings the *non-autonomous* DNLS lattice (2.41) to the *autonomous* form

$$\mathrm{i}\partial_z A_n + i\delta A_n + \epsilon_n (A_{n+1} - 2A_n + A_{n-1}) - \omega A_n + |A_n|^2 A_n = g_n, \qquad \omega > 0. \tag{2.42}$$

Among other interesting various boundary conditions which can be applied (periodic or Dirichlet), a case leading to the study of an infinite-dimensional system is the one-sided Dirichlet boundary condition $A_0 = 0$ on $n = 0$, see remark 2.7. Restricted to the case $\epsilon_n > 0$

for all $n \in \mathbb{Z}^+$ and $\epsilon_n \sim n$ (the simplest case is $\epsilon_n = n$ [29]), the natural phase space for the study of (2.42) supplemented with the one-sided Dirichlet boundary condition is the space $\ell^2_W(\mathbb{Z}_+)$ where $\mathbb{Z}^+ = \{1, 2, \ldots\}$ and $W_n = \epsilon_n^{-1} \sim n^{-1}$: the DNLS system (2.42) has the obvious energy equation

$$\frac{1}{2}\frac{d}{dz}\|A\|^2_{\ell^2_W} + \delta\|A\|^2_{\ell^2_W} = \mathrm{Im}\sum_{n=1}^{\infty} W_n \overline{A}_n g_n. \tag{2.43}$$

Based on the energy equation (2.43), we can apply the energy method (Ball [4, 5]) as in [24] to prove

Theorem 2.9. *Let the initial condition be $A^0 = \{A_n(0)\}_{n\in\mathbb{Z}^+}$ and $g \in \ell^2_W$. The dynamical system defined by (2.42), $\mathbf{S}(t) : \ell^2_W \to \ell^2_W$, $A(t) = \mathcal{S}(t)A^0$, possesses a global attractor in the strong topology of ℓ^2_W.*

Solutions in ℓ^2_W seem to be of physical significance since their possible slower rate of decay (and thus their possible different type of localization, for example, with a decaying oscillatory part) could be connected with the behavior of the localized solutions described in [29, pg 750-751].
Regarding now the operator

$$(\mathcal{L}_\omega A)_{n\in\mathbb{Z}^+} = -\epsilon_n(A_{n+1} - 2A_n + A_{n-1}) + \omega A_n, \tag{2.44}$$

is an one-dimensional analog of (2.12) and under some slight modifications, a similar result to theorem 2.4 can be shown: due to (2.6), for every $A \in \ell^2_{W^{-1}}$ we have

$$(\mathcal{L}_\omega A, A)_{\ell^2_W} = \sum_{n=1}^{\infty}|A_{n+1} - A_n|^2 + \omega\sum_{n=1}^{\infty}\frac{1}{W_n}|A_n|^2 \geqslant \nu_1\|A\|^2_{\ell^2_W}, \qquad \nu_1 = \frac{\omega}{\|W\|^2_\infty}. \tag{2.45}$$

The right-hand side of (2.45) defines the energy space $\mathcal{X}_E$ endowed with the norm $\|A\|^2_{\mathcal{X}_E} = (\mathcal{L}_\omega A, A)_{\ell^2_W}$. Using (2.8) we get the equivalence of norms

$$\omega\|A\|^2_{\ell^2_{W^{-1}}} \leqslant \|A\|^2_{\mathcal{X}_E} \leqslant (2\|W\|_\infty + \omega)\|A\|^2_{\ell^2_{W^{-1}}}, \tag{2.46}$$

thus $\mathcal{X}_E \equiv \ell^2_{W^{-1}}$. From lemma (2.2) and (2.9), the inclusion $\ell^2_{W^{-1}} \subset \ell^2_W$ is compact and $\mathcal{L}_\omega$ has a non-negative self-adjoint extension $\mathcal{L}_\omega : D(\mathcal{L}_\omega) \subseteq \ell^2_W \to \ell^2_W$ (here $D(\mathcal{L}_\omega) \equiv \ell^2_{W^{-1}}$), with a discrete spectrum in the positive real line. This time we work with the kernel $\mathcal{H}_{m,n}(t)$ of the linear equation $\partial_t\mathcal{H}_{m,n}(t) + (\mathcal{L}_\omega\mathcal{H})_{n\in\mathbb{Z}^+} = 0$. Repeating the proof of theorem 2.4, we use instead of (2.30), the inequality

$$\left(\sum_{n=1}^{\infty}\mathcal{H}^{2(q-1)}_{m,n}(t)\right)^{\frac{1}{q-1}} \leqslant \frac{1}{\nu_1}\left[\sum_{n=1}^{\infty}|\mathcal{H}_{n+1,m}(t) - \mathcal{H}_{n,m}(t)|^2 + \sum_{n=1}^{\infty}\frac{1}{W_n}\mathcal{H}^2_{m,n}(t)\right],$$

which comes out if one combines (2.2), (2.9) and (2.46). For $W_n = \epsilon_n^{-1} \sim n^{-1}$, we have due to remark 2.7 the estimate

$$\lambda_j \geqslant \frac{\nu_1}{e}\left(\sum_{n\in\mathbb{Z}^+} W_n^{\frac{q-1}{q-2}}\right)^{-\frac{q-2}{q-1}} j^{\frac{q-2}{q-1}}, \qquad j \to \infty, \quad \text{for all} \quad q \geqslant 3. \tag{2.47}$$

In the simplest case $W_n = \epsilon_n^{-1} = n^{-1}$, we get

$$\lambda_j \geqslant \frac{\omega}{e}\left[\zeta\left(\frac{q-1}{q-2}\right)\right]^{-\frac{q-2}{q-1}} j^{\frac{q-2}{q-1}}, \qquad j \to \infty, \quad \text{for all} \quad q \geqslant 3.$$

The estimate (2.47) could be useful in the study of the transformation of the m-dimensional volumes in ℓ^2_W from the linearized flow associated with (2.42) (see Ghidaglia [20, section 3.2, pg 400]), in order to prove that the global attractor is finite dimensional. For instance, estimates of the form (2.47) constitute an essential step for the derivation of lower bounds on the m-sum of the eigenvalues of the linear operator involved in the nonlinear evolution equation. With these lower bounds in hand, one can proceed to the application of Constantin–Foias–Temam theory for the investigation of the possible exponential decay of the m-dimensional volume carried out by the linearized flow, [36, lemma VI.2.1, pg 390, proposition 2.1, pg 364 and theorem 3.3, pg 374]. This task is not trivial and its completion will be considered in a subsequent work.

We conclude by noting that a continuous case of the weighted operators studied in this work appears in the study of equations of the form $-|x|^2\Delta u = \lambda u + f(u)$, related to the Wheeler–DeWitt quantum cosmological models, Berestycki and Esteban [7], Esteban and Giacomoni [17].

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